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Best L^1-Approximation by Polynomials, II

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1. INTRODUCTION

Let P_n denote the set of all real polynomials of degree not exceeding n and put (for any continuous f: $[-1, 1] \to \mathbb{R}$)

$$E_n(f) = \inf_{p \in P_n} \int_{-1}^{1} |f(t) - p(t)| \, dt, \qquad n = 0, 1, 2, \ldots.$$

f is said to be in the Markoff class M_n, provided that $(f - p_f) U_n$ does not change sign on $[-1, 1]$, where $U_n(t)$ denotes the nth Chebyshev polynomial of the second kind and p_f denotes the interpolation polynomial of f with respect to the zeros of U_n.

It is well known ([2, p. 94] or [4, p. 274, Theorem 2]) that

$$E_{n-1}(f) = \left| \int_{-1}^{1} f(t) \operatorname{sgn} U_n(t) \, dt \right| \qquad \text{for any} \quad f \in M_n \ (n \geqslant 1).$$

H. Brass [3] points out that

$$\int_{-1}^{1} f(t) \operatorname{sgn} U_n(t) \, dt = 2 \cdot \sum_{k=0}^{\infty} \frac{b_{(2k+1)(n+1)-1}}{2k+1} \tag{1.1}$$

provided that the expansion

$$f \sim \sum_{k=0}^{\infty} b_k U_k$$

is known. He remarks further that in many known examples the coefficients b_k tend rapidly to zero and that in these cases $2 |b_n| = 2 |b_n(f)|$ yields an asymptotic expansion of $E_{n-1}(f)$ (all of his examples lie in the Markoff-class M_n).

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We derive in this paper upper and lower estimates for $E_{n-1}(f)-2b_n$ which are applicable even if f does not lie in the Markoff class M_n, such as

$$f(t)=\cos(\omega t) \qquad \text{with} \quad |\omega|>\pi/2.$$

2. Notations and Lower Estimates

We have

$$2b_n(f)=(4/\pi)\cdot\int_{-1}^{1} f(t)\,U_n(t)(1-t^2)^{1/2}\,dt. \tag{2.1}$$

If f is an n-fold integral of a real integrable function then [4, p. 270, formula (1.2)]

$$\int_{-1}^{1} f(t)\operatorname{sgn} U_n(t)\,dt=\int_{-1}^{1} V_n(t)\,f^{(n)}(t)\,dt, \tag{2.2}$$

where the kernel $V_n(t)$ is given in [4, p. 276, Theorem 4]. From [4, p. 279, Theorem 5 and p. 270, formula (1.5)] we have

$$|V_n(t)|\leqslant 2^{3-n}(1-t^2)^n n^{1/2}/n! \tag{2.3}$$

for natural n and $-1\leqslant t\leqslant 1$.

Proposition ($n\in\mathbb{N}, f\in L^1[-1,1], b_j=b_j(f)$).

$$2\left|\sum_{k=1}^{\infty}\frac{b_{(2k+1)(n+1)-1}}{2k+1}\right|\leqslant 3E_{3n+1}(f). \tag{2.4}$$

Proof. Define the c_j by

$$E_{3n+1}(f)=\left\|f-\sum_{k=0}^{3n+1}c_k U_k\right\|_1.$$

From (1.1) we obtain

$$E_{3n+1}(f)\geqslant\left|\int_{-1}^{1}\left(f-\sum_{k=0}^{3n+1}c_k U_k\right)\operatorname{sgn} U_n\,dt\right|$$
$$=\left|2(b_n-c_n)+2\sum_{k=1}^{\infty}\frac{b_{(2k+1)(n+1)-1}}{2k+1}\right|.$$

The proposition follows since (observe (2.1) and $|U_n(t)(1-t^2)^{1/2}|\leqslant 1$)

$$2\,|b_n-c_n| = (4/\pi)\left|\int_{-1}^{1}\left(f(t)-\sum_{k=0}^{3n+1} c_k U_k(t)\right) U_n(t)(1-t^2)^{1/2}\,dt\right|$$

$$\leqslant (4/\pi)\left\|f-\sum_{k=0}^{3n+1} c_k U_k\right\|_1 = (4/\pi)\, E_{3n+1}(f). \quad ▮$$

Our proposition yields immediately the following general lower estimate:

COROLLARY ($n\in\mathbb{N}, f\in L^1[-1,1]$).

$$E_{n-1}(f)\geqslant 2\left|\sum_{k=0}^{\infty}\frac{b_{(2k+1)(n+1)-1}}{2k+1}\right|\geqslant 2\,|b_n|-3E_{3n+1}(f).$$

Proof. The first inequality is an immediate consequence of [4, p. 274, Theorem 2a] and of (1.1) and the second inequality follows from the proposition. ▮

Remark. The lower estimate as given in the corollary is in general better than the trivial estimate (let $p\in P_n$ yield the best approximation)

$$E_{n-1}(f)=\int_{-1}^{1}|f(t)-p(t)|\,dt$$

$$\geqslant\left|\int_{-1}^{1}\{f(t)-p(t)\}\,U_n(t)(1-t^2)^{1/2}\,dt\right|$$

$$=\left|\int_{-1}^{1} f(t)\,U_n(t)(1-t^2)^{1/2}\,dt\right| = (\pi/2)\,|b_n(f)|.$$

Here we used (2.1) and the fact that $U_n(t)(1-t^2)^{1/2}$ is orthogonal to P_{n-1}.

If $b_m\geqslant 0$ for $m\geqslant n$ then the corollary implies $E_{n-1}(f)\geqslant 2b_n(f)$ without any error term. This is the case, e.g., if $f(t)=\sum_{k=0}^{\infty} a_k t^k$ on $[-1,1]$ with $a_k\geqslant 0$ for $k\geqslant n$, since always $b_m(t^k)\geqslant 0$ (see [5, p. 100]).

3. THE MAIN RESULT

One knows that f belongs to the Markoff class M_n, provided that $f^{(n)}\geqslant 0$ on $[-1,1]$, and then (1.1) is applicable for handling $E_{n-1}(f)$. If $f^{(n)}\geqslant 0$ holds true only in a neighbourhood of 0, then f will lie outside M_n in general. Nevertheless we have the following result.

THEOREM ($n\in\mathbb{N}; f\in C^n[-1,1]; f^{(n)}(x)\geqslant 0$ *for* $|x|\leqslant\delta\leqslant 1$).

$$|E_{n-1}(f)-2b_n(f)|\leqslant\frac{2^{4-n}n^{1/2}}{n!}\cdot\int_{\delta\leqslant|t|\leqslant 1}(1-t^2)^n\,|f^{(n)}(t)|\,dt$$
$$+2\left|\sum_{k=1}^{\infty}\frac{b_{(2k+1)(n+1)-1}}{2k+1}\right|.$$

Proof. We use [4, p. 274, Theorem 2d] and get

$$\begin{aligned}E_{n-1}(f)&\leqslant\int_{-1}^{1}V_n(t)\,|f^{(n)}(t)|\,dt\\&\leqslant 2\int_{\delta\leqslant|t|\leqslant 1}V_n(t)\,|f^{(n)}(t)|\,dt+\int_{-1}^{1}V_n(t)\,f^{(n)}(t)\,dt\\&=\mathrm{I}+\mathrm{II}.\end{aligned}\tag{3.1}$$

Now (2.2) and (1.1) yield

$$\mathrm{II}=2\cdot\sum_{k=0}^{\infty}\frac{b_{(2k+1)(n+1)-1}}{2k+1}.$$

Thus the theorem follows from (2.3). ∎

If we use our proposition (2.4) we obtain

COROLLARY 1 ($n\in\mathbb{N}; f\in C^n[-1,1]; f^{(n)}(x)\geqslant 0$ *for* $|x|\leqslant\delta\leqslant 1$).

$$\begin{aligned}&|E_{n-1}(f)-2b_n(f)|\\&\quad\leqslant\frac{2^{4-n}n^{1/2}}{n!}\int_{\delta\leqslant|t|\leqslant 1}(1-t^2)^n\,|f^{(n)}(t)|\,dt+3E_{3n+1}(f).\end{aligned}$$

Applying (3.1) and (2.3) we get

COROLLARY 2 ($n\in\mathbb{N}; f\in C^{3n+2}[-1,1]; f^{(n)}(x)\geqslant 0$ *for* $|x|\leqslant\delta\leqslant 1$).

$$|E_{n-1}(f)-2b_n(f)|\leqslant\frac{2^{4-n}n^{1/2}}{n!}\int_{\delta\leqslant|t|\leqslant 1}(1-t^2)^n\,|f^{(n)}(t)|\,dt$$
$$+\frac{2^{4-3n}n^{1/2}}{(3n+2)!}\int_{-1}^{1}(1-t^2)^{3n+2}\,|f^{(3n+2)}(t)|\,dt.$$

4. Applications

Let $f(x)=(-1)^n \cos \omega x$. The case $|\omega| \leqslant \pi/2$ was treated in [3]. Let $|\omega| > \pi/2$ and put $\delta=(\pi/2)/|\omega|$ and $q=1-\delta^2$. Corollary 1 implies that (replace n by $2n$; $n \in \mathbb{N}$)

$$|E_{2n-1}(f)-2b_{2n}(f)| \leqslant \frac{2^{6-2n} n^{1/2}}{(2n)!} q^{2n} \omega^{2n} + 3E_{6n+1}(f).$$

The right-hand side is exponentially smaller than $E_{2n-1}(f)$, since [4, p. 290]

$$E_{2n-2}(f)=E_{2n-1}(f)=\frac{2^{1-2n}\omega^{2n}}{(2n)!}\{1+O_\omega(1/n)\}.$$

It follows that

$$E_{2n-2}(f)=E_{2n-1}(f)=2b_{2n}(f)\cdot\{1+O_\omega(n^{1/2}q^{2n})\}. \tag{4.1}$$

Now use [5, p. 47] and the formula $2b_n=a_n-a_{n+2}$ from [3] with

$$a_n=\frac{2}{\pi}\int_{-1}^{1} f(x)\, T_n(x)(1-x^2)^{-1/2}\, dx$$

to obtain

$$2b_{2n}(f)=2\{J_{2n}(\omega)-J_{2n+2}(\omega)\}$$

with [5, p. 48]

$$J_k(\omega)=\frac{\omega^k}{2^k}\sum_{r=0}^{\infty}\frac{(-\omega^2)^r}{4^r r!\,(k+r)!}.$$

So, in the present case, b_{2n} is explicitly known and can easily be expanded into an asymptotic series. In other cases it might be useful to remember the formula (see [1, p. 785, #22.11.4])

$$2b_n(f)=\int_{-1}^{1} V_n^*(t)\, f^{(n)}(t)\, dt,$$

where

$$V_n^*(t)=\frac{2^{3/2}}{\pi}\frac{n+1}{n!}\gamma_n\left(\frac{1-t^2}{2}\right)^{n+1/2}$$

and

$$\gamma_n = \frac{\pi^{1/2}\Gamma(n+1)}{\Gamma(n+3/2)} \asymp n^{-1/2}.$$

As another application of (1.1) we remark that the above theorem on the asymptotic behaviour of $E_{n-1}(f)$ is connected with [4, p. 286, Theorem 7b] and [4, p. 287, Theorem 8b]. In order to get asymptotic results from these theorems, one needs the quantities

$$c_{nk} = \frac{2^{n-1}(n+2k)!}{(2k)!} \int_{-1}^{1} V_n(t)\, t^{2k}\, dt \qquad \text{for} \quad n \geqslant 1 \text{ and } k \geqslant 0$$

explicitly. Applying (2.2) to $f(t) = t^{n+2k}$ and using (1.1) afterwards yields

$$\begin{aligned} c_{nk} &= 2^{n-1} \int_{-1}^{1} t^{n+2k} \operatorname{sgn} U_n(t)\, dt \\ &= 2^n \cdot \sum_{r=0}^{\infty} \frac{b_{(2r+1)(n+1)-1}(f)}{2r+1} \\ &= \frac{(n+1)}{2^{2k}} \sum_{r=0}^{[k/(n+1)]} \frac{1}{(r+1)(n+1)+k} \binom{n+2k}{k-r(n+1)}, \end{aligned}$$

where the last equation is a consequence of [5, p. 100]. In particular we have

$$c_{nk} = \frac{(n+1)(n+2k)!}{2^{2k} k!\, (n+1+k)!} \qquad \text{for} \quad 0 \leqslant k < n+1,$$

which was mentioned to us in a letter by H. Brass.

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